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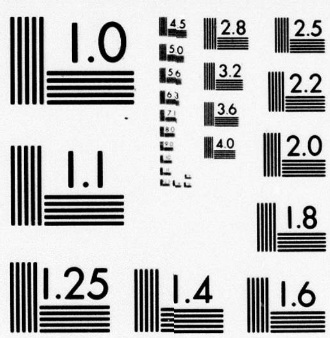

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6 PARETO EFFICIENCY

WITH COSTLY TRANSFERS.

by

10 Kenneth J. Arrow

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Harvard University
Littauer #309
Cambridge, Mass. 02138

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PARETO EFFICIENCY WITH COSTLY TRANSFERS

Kenneth J. Arrow

This paper is dedicated to the distinguished Polish economist, Edward Lipiński, scholar and man of moral integrity and courage, on his ninetieth birthday.

0. Introduction and Summary.

The theoretical notion of Pareto efficiency has been an important clarifying concept in comparing alternative resource allocations, both in theory and in the formation of economic policy. In particular, the close link between Pareto efficiency and competitive equilibrium is the central result for both analysis and policy. The equivalence of the two concepts is stated in the form of two theorems:

First Theorem of Welfare Economics. Every competitive equilibrium is Pareto efficient.

Second Theorem of Welfare Economics. For every Pareto efficient allocation of resources, there is a redistribution of the endowments such that the given Pareto efficient allocation is a competitive equilibrium for the new endowment distribution.

The Second Theorem in particular implies that problems of equity can be separated from those of efficiency; if the existing distribution of welfare is judged inequitable, rectification should proceed by redistributing endowments ("lump-sum transfers") and then allowing the market to work unimpeded rather than by direct interference with the market in the form, say, of price controls or rationing.

I have not stated the well-known hypotheses for the validity of the two Theorems; these hypotheses, roughly the existence of all relevant markets (including those for externalities) and convexity, at least on the production side, are of course frequently violated, and the theory of government policy is an attempt to suggest one class of remedies.

This literature is vast, and I will not trod that ground here.

Another objection to the application of the two Theorems is also widely known but its analysis is not yet well explored. I refer to the impossibility of distributing the endowments without some cost. Apart from poll taxes, which certainly have no appeal as instruments for achieving equity, we have no effective means of transfering endowments from one individual to another without some loss due to incentives. Any tax that is proposed will usually fall on some margin of the individual's choice and cause a price distortion. The redistribution itself, then, will cause an inefficiency; even if the market is allowed to operate without impedance after the transfers, the final state of the system will be inefficient.

Once it is recognized that redistributive transfers are costly, the concept of Pareto efficiency needs modification to take account of losses during the redistribution process. Hence, whether a given allocation is Pareto efficient or not will in general depend on the amount of transferring needed to achieve it and therefore on the initial distribution of endowments.

In this paper, I seek to initiate a general discussion of Pareto efficiency and its relation to competitive equilibrium when transfers are costly. For simplicity, I confine discussion to a pure exchange economy (no production).

It is necessary to specify a <u>transfer technology</u>, a concept that has already appeared in the literature in connection with competitive equilibrium in the work of Foley [1970], Hahn [1971], and others. Here, it is applied to transfers through extra-market means (primarily government compulsion) as well as through the market. For this initial

paper, I will make the simplest possible assumption, that the losses in transfering a given commodity are in terms of that commodity and proportional to the transfer.

In this model, it is then straightforward to characterize allocations which are Pareto efficient relative to a given endowment allocation. We can define an allocation as being Pareto efficient without qualification if it is Pareto efficient for <u>some</u> endowment allocation. The class of Pareto efficient allocations can be characterized in an interesting way in terms of a cycle condition, that a sequence of pairwise trades between successive elements of a closed cycle of economic agents not be advantageous.

It is easy to demonstrate that if the market transfer technology is the same as the redistributive transfer technology, then the First Theorem of Welfare Economics remains valid. However, the second is clearly false, so that the trade-off between efficiency and equity becomes unavoidable.

1. <u>Allocations Pareto Efficient with Respect to Initial Endowments.</u>

Notation:

x_k^i = amount of commodity k used by indidual i;

x = allocation of commodities to individuals, with components x_k^i;

x^i = commodity vector of individual i, i.e., with components x_k^i for fixed i;

ω_k^i = amount of commodity k in individual i's endowment;

ω = endowment allocation of commodities to individuals, with components ω_k^i;

ω^i = endowment commodity vector of individual i;

$U^i(x^i)$ = utility of individual i from commodity vector x^i. I assume U^i differentiable.

U^i_k = $\partial U^i / \partial x^i_k$.

u^i = commodity vector of withdrawals from individual i;

u = allocation of withdrawals, with components u^i_k;

v^i = commodity vector of transfers to individual i.

v = allocation of transfers to individuals, with components v^i_k;

The two transfer vectors u^i and v^i are taken to be non- negative. From the notation, final and endowment allocations are related by,

$$x^i = \omega^i - u^i + v^i, \text{ all } i. \tag{1}$$

We will have to require that $x^i \geqq 0$,

Definition 1. The set of admissible pairs (u, v) of allocations of withdrawals and transfers is termed the transfer technology, T.

Under the usual assumptions of costless transfer, the transfer technology is defined by the conditions,

$$u^i \geqq 0, \; v^i \geqq 0, \text{ all } i; \; \sum_i u^i \geqq \sum_i v^i,$$

Definition 2. The transfer technology is said to be simple if, for each k, there exists a parameter, β_k, such that the transfer technology T is defined by the relations,

$$u^i \geqq 0, \; v^i \geqq 0, \text{ all } i; \tag{2}$$

$$\sum_i v^i_k \leqq \beta_k \sum_i u^i_k, \text{ all } k. \tag{3}$$

The parameter, β_k, is the proportion of goods taken from some individuals which is still available to be given to others. We assume, of course, that

$$0 < \beta_k < 1, \tag{4}$$

so that transfer is possible but with some possible loss. The loss may differ among commodities.

(In a more general transfer technology, the transfer of one good will be at the expense of other goods rather than itself. The present case is treated only because of its simplicity.)

Definition 3. The allocation x is said to be attainable from ω if there exists a pair of withdrawal and transfer allocations, (u, v) belonging to the transfer technology for which (1) holds with,

$$x^i \geq 0 . \tag{5}$$

Definition 4. The allocation x is said to be Pareto efficient with respect to the endowment allocation ω if x is attainable from ω and if there does not exist x' attainable from ω such that $U_i(x'^i) \geq U_i(x^i)$, all i, $U_j(x'^j) > U_j(x^j)$, some j.

Let,

$$A(\omega) = \{x \mid x \text{ attainable from } \omega\}.$$

Since T is defined in a way independent of ω, we have,

$$A(\omega) = (\{\omega\} + T) \cap X^t, \tag{6}$$

where $\{\omega\}$ is the set consisting of ω alone, and $X^t = \{x \mid x \geq 0\}$.

Pareto efficiency with respect to ω means simply Pareto efficiency over $A(\omega)$. Then, if the utility functions are quasi-concave and satisfy some additional regularity properties, x is Pareto efficient over a convex set if an only if there exist non-negative multipliers, λ_i, not all zero, such that x maximizes,

$$\sum_i \lambda_i U^i(x^i), \tag{7}$$

over that set. In view of the structure of $A(\omega)$, x is Pareto efficient

over $A(\omega)$ if there exist $\lambda_i \geq 0$, not all zero, u^i, v^i, which maximize (7) subject to (2), (3) and (5), with x^i defined by (1).

Let p_k be the Lagrange multiplier associated with the constraint (3) and q_{ik} that associated with the constraint, $x_k^i \geq 0$, (5). The Lagrangian can then be written,

$$L = \sum_i \lambda_i U^i(\omega^i - u^i + v^i) + \sum_k p_k(\beta_k \sum_i u_k^i - \sum_i v_k^i)$$

$$+ \sum_i \sum_k q_{ik} (\omega_k^i - u_k^i + v_k^i). \tag{8}$$

Since the variables u_k^i, v_k^i are constrained to be non-negative, necessary conditions for an optimal allocation are that, for all i and k,

$$\partial L/\partial u_k^i \leq 0, \text{ with equality if } u_k^i > 0, \tag{9a}$$

$$\partial L/\partial v_k^i \leq 0, \text{ with equality if } v_k^i > 0. \tag{9b}$$

In addition, the inequalities (3) and (5) must hold; the corresponding Lagrange parameters must be non-negative, and, if any are positive, the corresponding inequality must become an equality. These conditions, together with (9), constitute a system of linear equations and inequalities in the Lagrange parameters for a given transfer; the solvability of this system is equivalent to the Pareto optimality (with respect to the initial endowment ω) for the allocation x defined by the given transfers u, v according to (1).

We now write out the sytem of inequalities explicitly. From (9a) and (8),

$$-\lambda_i U_k^i + \beta_k p_k - q_{ik} \leq 0, \text{ all i and k},$$

or,

$$\lambda_i U_k^i + q_{ik} \geq \beta_k p_k, \text{ for all i and k}; \tag{10a}$$

and,

$$\lambda_i U^i_k + q_{ik} = \beta_k p_k \text{ if } u^i_k > 0. \tag{10b}$$

Similarly, if we replace i by j in (9b), we find,

$$\lambda_j U^j_k + q_{jk} \leqq p_k, \text{ all j and k;} \tag{11a}$$

$$\lambda_j U^j_k + q_{jk} = p_k \text{ if } v^j_k > 0. \tag{11b}$$

The inequalities on the Lagrange parameters are,

$$\lambda_i \geqq 0, \text{ all i;} \tag{12a}$$

$$\lambda_j > 0, \text{ some j.} \tag{12b}$$

$$p_k \geqq 0, \text{ all k.} \tag{13a}$$

$$q_{ik} \geqq 0, \text{ all i and k.} \tag{14a}$$

If $p_k > 0$, then constraint (3) must hold with equality for the corresponding k.

$$\sum_i v^i_k = \beta_k \sum_i u^i_k \text{ if } p_k > 0. \tag{13b}$$

Similarly, $q_{ik} > 0$ implies that constraint (5) must hold with equality for the corresponding i and k. In the contrapositive form, this statement reads,

$$q_{ik} = 0 \text{ if } x^i_k > 0. \tag{14b}$$

Note that the marginal utilities U^i_k and U^j_k in (10) and (11) are evaluated at x^i, as defined by (1).

Let us postulate that there is no satiation in any good. That is,

$$U^i_k > 0, \text{ all i and k, for all } x^i. \tag{15}$$

From (12b) and (15), $\lambda_j U^j_k > 0$, for all k, for some j. Since $q_{jk} \geqq 0$, by (14a), it follows from (11a) that $p_k > 0$, for all k.

Hence, (13a, b) can be rewritten,

$$p_k > 0, \text{ all } k, \tag{16a}$$

$$\sum_i v_k^i = \beta_k \sum_i u_k^i \text{ for all } k. \tag{16b}$$

Suppose for some i and k, we had both $u_k^i > 0$ and $v_k^i > 0$ (i.e., an individual both gave andreceived commodity k). Then from (10b) and (11b) (replacing j by i in the latter), we must have $p_k = \beta_k\ p_k$. But this is impossible, since $p_k > 0$ by (16a) and $\beta_k < 1$, by assumption (4).

For all i and k, it cannot be that both $u_k^i = 0$ and $v_k^i = 0$ (17)

If we review the system of inequalities (2), (10), (11), (12), (14), and (16), we observe first that the endowments ω do not appear explicitly. The primal variables appearing explicitly are x, u, and v. These determine ω, for, from (1),

$$\omega^i = x^i + u^i - v^i. \tag{18}$$

This suggests that a natural rephrasing of the original question is to start with a given x, u, v, and ask whether x is Pareto efficient for the corresponding ω. In a still further rephrasing, we can start with a (final) allocation x and ask for the set of endowments ω such that x is Pareto efficient with respect to ω. (This set may of course be empty.) This is equivalent to seeking the solution of the system in the Lagrange parameters and the variables u, v.

Since the endowment allocation must be non-negative, it follows from (18) that the transfers u, v must satisfy,

$$x^i + u^i - v^i \geqq 0, \text{ all } i. \tag{19}$$

It is also to be observed that, except for the equation (16b) and the inequalities (2), the variables u, v enter only through their signs

(in (10b) and (11b)). Consider first, then, the remainder of the system, i.e., (10a), (11a), (12), (14), and (16a). For fixed x, the coefficients U_k^i and U_k^j are given. Call this the <u>inner</u> system. The variables are just the Lagrange parameters. The inner system may or may not be solvable. If it is not, then clearly x is not Pareto efficient for any endowment ω. If it is, take any solution. Rewrite (10b) and (11b) in contrapositive form:

$$u_k^i = 0 \text{ if } \lambda_i U_k^i + q_{ik} > \beta_k p_k, \tag{20}$$

$$v_k^j = 0 \text{ if } \lambda_j U_k^j + q_{jk} < p_k. \tag{21}$$

Then, given the Lagrange parameters which solve the inner system, we have a system of equations and inequalities in u, v, namely, (2), (16b), (19), (20), and (21), which may be termed the <u>outer system</u>. Note that this system always has at least one solution, namely, $u^i = v^i = 0$, for all i. In this case, we have $\omega = x$. Thus, if the inner system is solvable, then x is Pareto efficient with respect to itself.

However, for any given solution of the inner system, there are in general many solutions of the outer system. For each solution, there is a corresponding ω, defined by (18). More detailed properties of these solutions, and a useful necessary and sufficient condition for solvability of the inner system will be found in the following sections. In the meantime, the results found thus far can be summarized in the following definition and theorems.

<u>Definition 5.</u> The allocation x is said to be <u>Pareto efficient</u> (without qualification) if it is Pareto efficient with respect to some ω.

<u>Theorem 1.</u> The allocation x is Pareto efficient if and only if it is Pareto efficient with respect to itself. A necessary and sufficient condition that x be Pareto efficient is that the following system of

equations and inequalities have a solution in the variables, λ_i, p_k, q_{ik}:

$$\lambda_i U^i_k + q_{ik} \geqq \beta_k p_k, \tag{a}$$

$$\lambda_j U^j_k + q_{jk} \leqq p_k, \tag{b}$$

$$\lambda_i \geqq 0, \text{ all } i, \tag{c}$$

$$\lambda_j > 0, \text{ some } j, \tag{d}$$

$$q_{ik} \geqq 0, \tag{e}$$

$$q_{ik} = 0 \text{ if } x^i_k > 0, \tag{f}$$

$$p_k > 0. \tag{g}$$

Here, U^i_k is evaluated at x^i. The system (a-g) will be referred to as the <u>inner system</u> (for x).

<u>Theorem 2.</u> Let x be Pareto efficient. For any solution, λ_i, p_k, q_{ik} to the corresponding inner system, let u, v satisfy the following system of equations and inequalities:

$$u^i \geqq 0, \tag{a}$$

$$v^i \geqq 0, \tag{b}$$

$$\sum_i v^i_k = \beta_k \sum_i u^i_k, \tag{c}$$

$$x^i + u^i - v^i \geqq 0, \tag{d}$$

$$u^i_k = 0 \text{ if } a_{ik} > \beta_k, \tag{e}$$

$$v^i_k = 0 \text{ if } a_{ik} < 1, \tag{f}$$

where,

$$a_{ik} = (\lambda_i U^i_k + q_{ik}) / p_k \tag{g}$$

Then if $\omega^i = x^i + u^i - v^i$, x is Pareto efficient for ω. The system (a)-(f) will be referred to as the <u>outer system.</u>

It is interesting to note that the outer system depends on the solution of the inner system only through (e) and (f), which designate zero values for certain transfers.

2. <u>Simplification of the Outer System</u>.

For a given solution of the inner system, the outer system can be given a somewhat simplified form. Let,

$$\bar{a}_k = \max_i \underline{a}_{ik},\ \underline{a}_k = \min_i a_{ik}. \tag{22}$$

From Theorem 1(a) and (b), $\bar{a}_k \leq 1$, $\underline{a}_\beta \geq \beta_k$, so that $\underline{a}_k/\bar{a}_k \geq \beta_k$. Suppose the strict inequality holds. Then either $\underline{a}_k > \beta_k$ or $\bar{a}_k < 1$. In the first case, $a_{ik} > \beta_k$, all i, so that $u_k^i = 0$ for all i, by Theorem 2(e). Therefore, $\sum_i v_k^i = 0$, from Theorem 2(c); since $v^i \geq 0$ by Theorem 2(b), we must have $v_k^i = 0$ for all i. In the second case, $v_k^i = 0$ for all i, by Theorem 2(f), so that by corresponding reasoning $u_k^i = 0$ for all i. In either case,

$$\text{if } \underline{a}_k/\bar{a}_k > \beta_k, \text{ then } u_k^i = v_k^i = 0 \text{ for all i.} \tag{23}$$

Now suppose $\underline{a}_k/\bar{a}_k = \beta_k$. Then $\underline{a}_k = \beta_k$, $\bar{a}_k = 1$. From the definitions (22), this means there is at least one individual i for whom a_{ik} takes on its least possible value, β_k, and at least one for whom it takes on its greatest possible value, 1. Let,

$$\underline{S}_k = \{i \mid a_{ik} = \beta_k\},\ \bar{S}_k = \{i \mid a_{ik} = 1\}. \tag{24}$$

From Theorem 2(e-f),

$$u_k^i = 0 \text{ if } i \notin \underline{S}_k,\ v_k^i = 0 \text{ if } i \notin \bar{S}_k. \tag{25}$$

Theorem 2(c) now becomes,

$$\sum_{i \varepsilon \overline{S}_k} v_k^i = \beta_k \sum_{i \varepsilon \underline{S}_k} u_k^i. \tag{26}$$

If $i \notin \overline{S}_k$, then Theorem 2(d) reduces to the statement, $x_k^i + u_k^i \geq 0$, which is automatically satisfied. For $i \varepsilon \overline{S}_k$, Theorem 2(d, b) become the statement,

$$0 \leq v_k^i \leq x_k^i \text{ for } i \varepsilon \overline{S}_k. \tag{27}$$

If we refer back to the definition of ω^i in (18) and make use of (25), we see that,

$$u_k^i = \omega_k^i - x_k^i \text{ for } i \varepsilon \overline{S}_k,\ v_k^i = x_k^i - \omega_k^i \text{ for } i \varepsilon \underline{S}_k . \tag{28}$$

Formulas (23) - (28) together can be restated as the following theorem.

Theorem 3. Let x be Pareto efficient, and let T(x) be the set of solutions (λ_i, p_k, q_{ik}) to the inner system for x. For any element of T(x), let,

$$a_{ik} = (\lambda_i U_k^i + q_{ik})/p_k, \tag{a}$$

$$\underline{S}_k = \{i \mid a_{ik} = \beta_k\},\ \overline{S}_k = \{i \mid a_{ik} = 1\} . \tag{b}$$

Define $\Omega(x, \lambda_i, p_k, q_{ik})$ to be the set of endowment allocations ω satisfying the following conditions (c-e):

for any commodity k for which $\min_i a_{ik} > \beta_k \max_i a_{ik}$, $\omega_k^i = x_k^i$

for all individuals i; (c)

for all other commodities,

$\omega_k^i \geq x_k^i$ for $i \varepsilon \overline{S}_k$, $0 \leq \omega_k^i \leq x_k^i$ for $i \varepsilon \overline{S}_k$,

$\omega_k^i = x_k^i$ if i belongs to neither $\underline{S}_k$ not $\overline{S}_k$; (d)

$$\sum_{i \in \underline{S}_k} \omega_k^i + \beta_k \sum_{i \in \overline{S}_k} \omega_k^i = \sum_{i \in \underline{S}_k} x_k^i + \beta_k \sum_{i \in \overline{S}_k} x_k^i. \qquad \text{(e)}$$

Then x is Pareto efficient for ω if and only if $\omega \in \Omega\ (x, \lambda_i, p_k, q_{ik})$ for some solution $(\lambda_i, p_k, q_{ik}) \in T(x)$.

3. Simplification of the Inner System and a Criterion for Pareto Efficiency.

We now analyze the inequality system of Theorem 1. In particular, it can be reduced to a system of inequalities in the utility weights, λ_i, alone. Since $q_{jk} \geqq 0$ by Theorem 1(e),

$$\lambda_j U_k^j \leqq p_k, \text{ all } j \text{ and } k. \qquad (29)$$

From Theorem 1(a, f),

$$\lambda_i U_k^i \geqq \beta_k p_k \text{ if } x_k^i > 0. \qquad (30)$$

It will be useful to distinguish those individuals, if any, for whom $x^i = 0$. These individuals are excluded in effect from all goods. In particular, therefore, (30) does not apply to them for any commodity k.

$$E = \{i \mid x^i = 0\}. \qquad (31)$$

If $i \notin E$, then $x_k^i > 0$, some k, and therefore, from (30),

$$\lambda_i > 0 \text{ if } i \notin E. \qquad (32)$$

If we assume that $j \notin E$, then we can divide (30) by (29) to find,

$$(\lambda_i U_k^i / \lambda_j U_k^j) \geqq \beta_k \text{ if } x_k^i > 0,\ j \notin E. \qquad (33)$$

Let Ω (x) be the projection of T(x) on the subspace of variables λ_i $(i \notin E)$, that is,

$$\Lambda(x) = \{\lambda_i,\ i \notin E \mid (\lambda_i, p, q_{ik}) \in T(x) \text{ for some } \lambda_i\ (i \in E), \text{ some } p_k, \text{ and some } q_{ik}\}. \qquad (34)$$

Then we have shown that any element of Λ (x) satisfies (33). Conversely, however, we shall show that for any solution of (33), with $\lambda_i > 0$, all $i \notin E$, we can find λ_i ($i \in E$), p_k, q_{ik} such that the inner system is satisfied. For given λ_i ($i \notin E$), satisfying (33), we have to show that Theorem 1(a, b, e, f. g) can be satisfied. We exhibit such a solution, namely,

$$\lambda_i = 0 \text{ for } i \in E, \tag{35}$$

$$p_k = \max_{j \notin E} \lambda_j U_k^j, \tag{36}$$

$$q_{ik} = \max (\beta_k p_k - \lambda_i U_k^i, 0). \tag{37}$$

From (37), it is immediately obvious that Theorem 1 (e) holds; from (36), Theorem 1(g) is true. Suppose $x_k^i > 0$. From (33),

$$\lambda_i U_k^i \geq \beta_k \lambda_j U_k^j \text{ for all } j \notin E.$$

In particular, choose j to maximize $\lambda_j U_k^j$; from (36),

$$\lambda_i U_k^i \geq \beta_k p_k,$$

so that, from (37), $q_{ik} = 0$ when $x_k^i > 0$, verifying Theorem 1(f).

Add $\lambda_i U_k^i$ to both sides of (37).

$$\lambda_i U_k^i + q_{ik} = \max (\beta_k p_k, \lambda_i U_k^i). \tag{38}$$

It follows immediately that Theorem 1(a) is verified.

If $\lambda_i U_k^i > \beta_k p_k$, then $i \notin E$, from (35), and,

$$\lambda_i U_k^i + q_{ik} = \lambda_i U_k^i \leq p_k,$$

by (38) and (36). If $\lambda_i U_k^i \leq \beta_k p_k$, then,

$$\lambda_i U_k^i + q_{ik} = \beta_k p_k < p_k,$$

from (38) and the fact that $\beta_k < 1$. Thus Theorem 1(b) also holds, and we have verified that,

$$\Lambda(x) \text{ is characterized by (33).} \tag{39}$$

We now restate (33).

$$\lambda_i/\lambda_j \geqq \beta_k(U^j_k/U^i_k) \text{ if } x^i_k > 0,\ j \notin E. \tag{40}$$

Let,

$$K_i = \{ k \mid x^i_k > 0 \}. \tag{41}$$

If $i \notin E$, the K_i is non-empty. Since k appears only on the right-hand side of (40), the inequalities (40) can be expressed by replacing the right-hand side by its maximum over k.

$$\lambda_i/\lambda_j \geqq \max_{k_j \varepsilon K_j} \beta_k\ (U^j_k/U^i_k) \text{ if } i,\ j \notin E. \tag{42}$$

Now take logarithms of both sides of (42). Let,

$$\mu_i = \log \lambda_i, \tag{43}$$

$$b_{ij} = \log \max_{k\, \varepsilon K_i} \beta_k\ (U^j_k/U^i_k). \tag{44}$$

(42) becomes,

$$\mu_i - \mu_j \geqq b_{ij}. \tag{45}$$

The conditions for solvability of the system of linear inequalities (45) have already been obtained by Afriat [1963]. They are conditions on the numbers b_{ij}. To state them we need some new terminology.

By a chain σ of individuals of length n will be meant an assignment of an individual to each of the numbers 0,...,n; thus, $\sigma(r)$ is the individual numbered r in the chain. If the chain has length 1, its coefficient will be b_{ij} with $i = \sigma(0)$, $j = \sigma(1)$. For longer chains, the <u>chain</u> <u>coefficient</u> will be the sum of the coefficients of the

successive links. Thus,

$$v(\sigma) = \sum_{r=1}^{n} b_{\sigma(r-1),\ \sigma(r)},$$

is the chain coefficient for a chain σ of length n.

A particular kind of chain is a <u>cycle</u>, where the beginning and end of the chain are the same, i.e., where $\sigma(0) = \sigma(n)$, where n is the length of σ. Then Afriat has shown ([1963], Theorem 7.2, p. 131, slightly restated) that a necessary and sufficient condition for the solvability of (45) is that $v(\sigma) \leq 0$ for all cycles σ.

It is useful to interpret this condition. First of all, the term,

$$\max_{k \in K_i} \beta_k \,(U_k^j / U_k^i),$$

indicates the most efficient way of improving individual j's welfare by transferring from individual i. Let k(i, j) denote the commodity permitting of the most efficient transfer. Then, by using the definition of $v(\sigma)$ and taking antilogarithms, the condition that $v(\sigma) \leq 0$ becomes,

$$\prod_{r=1}^{n} \beta_{k(\sigma(r-1),\ \sigma(r))} \left[U^{\sigma(r)}_{k(\sigma(r-1),\ \sigma(r))} / U^{\sigma(r-1)}_{k(\sigma(r-1),\ \sigma(r))}\right] \leq 1 \quad (46)$$

If (46) were violated for a cycle σ, then there would be a successive sets of transfers around a cycle which would improve the lot of the initial individual and not hurt anyone else, a clear violation of Pareto efficiency.

<u>Theorem 4.</u> A necessary and sufficient condition that x be Pareto efficient is that condition (46) hold for any cycle of individuals. If it holds, then all solutions of the inner system of Theorem 1 can be obtained as follows: Let $E = \{i \mid x^i = 0\}$. Then find λ_i $(i \notin E)$ as the solutions of the system of inequalities,

$$\lambda_i/\lambda_j \geq \max_{k \in K_i} \beta_k \, (U^j_k/U^i_k), \; \lambda_i > 0, \text{ for } i, j \notin E. \qquad \text{(a)}$$

Then for any given solution of (a), choose λ_i ($i \in E$), p_k, q_{ik}, to satisfy Theorem 1 (a, b, e, f, g).

4. Pareto Efficiency and Competitive Equilibrium Under Costly Transfer.

Suppose there is a market, rather than direct redistribution. Suppose however the costs of transfer are the same, i.e., a sale of commodity k to the market permits purchases of a proportion of only β_k. Then buying and selling prices must be related correspondingly. The conditions for competitive equilibrium are obvious and coincide with those for Pareto efficiency.

Theorem 5. Suppose that in a *competitive* market only a fraction β_k of the sales of commodity k are available for purchase. Then a competitive equilibrium for a given endowment allocation ω is Pareto efficient for that endowment.

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13. ABSTRACT

(1) whether

The concept of Pareto efficiency, as ordinarily applied, implies that costless redistributive transfers are possible. This paper generalizes the concept to a simple case where transfers of a given good involve losses measurable in that good. The Pareto efficiency of a given allocation then depends on the initial distribution endowments. For a given allocation, then, we can ask: ~~(a) does~~ there exists any endowment allocation for which the given allocation is Pareto efficient ~~(b)~~ if there is, what is the class of endowment allocations for which it is efficient? These questions are answered in the paper.

; and (2)

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